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SENSITIVITY ANALYSIS FOR COUPLED THERMOELASTIC SYSTEMS

DANIEL A. TORTORELLI, G. SUBRAMANI and STEPHEN C. Y. LU
Department of Mechanical and Industrial Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.

and

ROBERT B. HABER
Department of Theoretical and Applied Mechanics and National Center for Supercomputing Applications, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.

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Abstract—This paper presents an adjoint approach, derived from the reciprocal theorem, for the sensitivity analysis of linear dynamic thermoelastic systems. The variation of a general response functional is expressed in explicit form with respect to variations of the design fields which consist of the material properties, applied loads, prescribed boundary, initial conditions, and the structural shape. The functional is dependent on these design quantities as well as the following implicity defined response fields: displacement, temperature, stress, strain, heat flux, temperature gradient, reaction forces, and reaction surface flux. The formulation incorporates the reciprocal relation between variations of the real system design and response fields and an adjoint state. Here, convolution is employed in lieu of time mappings used in other transient adjoint sensitivity derivations. Specializations of the formulation to uncoupled, combined quasi-static uncoupled, and steady-state thermoelasticity are also presented. The finite element method is used to demonstrate the application of the formulation to a problem in automobile engine design.

1. INTRODUCTION

One can define a general performance functional to characterize a system in terms of explicit design fields and implicit response fields. Design sensitivity analysis determines an explicit relationship between the variation of the design fields and the resulting variation in the performance functional. For a dynamic thermoelastic system, the explicit quantities in the response functional definition are the material properties, applied loads, prescribed boundary conditions, initial conditions, and the structural shape; while the implicit quantities include displacement, temperature, strain, temperature gradient, stress, heat flux vector, reaction force, and reaction surface flux fields. The design quantities and an initial-boundary-value problem governing the dynamic thermoelastic system implicitly determine these latter fields. Typically, the performance functional characterizes one or more design criteria for the system. For example, the functional might describe the maximum stress, mean stress, or stress amplitude at a point in the body during a load cycle, or the temperature and its gradient in a region of the body.

Design sensitivity analysis provides valuable information throughout the design process. When incorporated in a Taylor series expansion, the sensitivities estimate the performance of modified designs without additional analyses. Sensitivities are also integral parts of optimal design algorithms (Vanderplaats, 1984), identification studies (Flanigan, 1987), reliability analyses (Ang and Tang, 1975), and inverse problems (Beck *et al.*, 1985).

Design sensitivity analysis for elastic systems has been a subject of considerable interest during the past decade. For an extensive list of references see Olhoff and Taylor (1983), Haftka and Grandhi (1986), Choi *et al.* (1988) and Tortorelli *et al.* (1990). More recently, several papers appeared in which sensitivities for thermoelastic systems are derived. Dems and Mroz (1987) used an adjoint approach and direct differentiation to derive the design sensitivities for an uncoupled dynamic, thermoelastic system. They used a stress constitutive model which accommodated nonlinear elastic dependencies on strain and temperature. A linear isotropic model described the thermal constitutive relationship. They presented sensitivities for a general performance functional, dependent on strain, displacement, temperature, material properties, applied loads, and shape. Dems and Mroz used the material derivative concept (Haug *et al.*, 1986) to derive shape sensitivities. Meric (1986) derived the design sensitivities for linear, isotropic, steady-state thermoelastic systems using the Lagrange multiplier method (Belegundu, 1985). In this formulation, the design functional is dependent on stress, strain, heat flux, displacement, temperature, reaction forces, reaction surface flux, applied loads, and prescribed boundary conditions. Meric (1988) obtained shape sensitivities for dynamically-loaded, nonlocal thermoelastic solids using the Lagrange multiplier and material derivative methods. Shape sensitivities for nonlinear, dynamic, uncoupled thermoelastic systems are presented in Tortorelli *et al.* (1989).

Direct differentiation sensitivity analysis methods require the solution of a distinct pseudo problem to determine the derivatives of the response fields with respect to each design parameter. The chain rule is then applied to the response derivatives to evaluate the sensitivities of the performance functionals. In the adjoint method, the sensitivities are evaluated directly after an adjoint problem is solved for each functional. Thus, if the number of design parameters exceeds the number of performance functionals, then the adjoint method is preferred because it requires fewer solutions. If the number of performance functionals is large or the sensitivities of the complete response fields are required, then the direct differentiation approach is preferred. This paper pursues the adjoint approach.

The following three sections present the explicit sensitivity analysis for a general performance functional defined over a dynamic thermoelastic system. The sensitivity formulation uses reciprocity between load and response variations of the real load system and load adjoint system (Dems and Mroz, 1983). The convolution (Tortorelli *et al.*, 1990) replaces time mappings used in previous transient adjoint sensitivity analyses (Tortorelli and Haber, 1989). Domain parameterization (Haber, 1987; Phelan and Haber, 1988), an alternative to the material derivative method, is used to derive explicit sensitivities with respect to shape variations. This methodology offers an alternative to previous adjoint sensitivity formulations and extends them by considering the fully-coupled problem. Sensitivities for the uncoupled theory, combined quasi-static uncoupled theory, and steady-state theory are given in Section 5. In Section 6, a finite element implementation of the formulation illustrates the application of the methodology to automobile engine design.

2. SENSITIVITY PROBLEM AND GOVERNING EQUATIONS

Consider the general performance functional which characterizes some design criterion of a dynamic thermoelastic system:

$$G = \int_0^t \left[\int_B f(u_i, E_{ij}, S_{ij}, \vartheta, g_i, q_i, C_{ijkl}, \rho, K_{ij}, c, M_{ij}, \theta_0, b_i, r)\, dv + \int_{\partial B} g(u_i, s_i, \vartheta, q^s, h, \vartheta_\infty)\, da \right] d\tau. \quad (1)$$

The response fields consist of the displacement vector $\mathbf{u}(\mathbf{x}, \tau)$, infinitesimal strain tensor $\mathbf{E}(\mathbf{x}, \tau)$, Cauchy stress tensor $\mathbf{S}(\mathbf{x}, \tau)$, relative temperature $\vartheta(\mathbf{x}, \tau)$, temperature gradient $\mathbf{g}(\mathbf{x}, \tau)$, heat flux vector $\mathbf{q}(\mathbf{x}, \tau)$, surface traction $\mathbf{s}(\mathbf{x}, \tau)$, and surface flux $q^s(\mathbf{x}, \tau)$.† The symmetric elasticity tensor $\mathbf{C}(\mathbf{x})$, density $\rho(\mathbf{x})$, symmetric conductivity tensor $\mathbf{K}(\mathbf{x})$, specific heat $c(\mathbf{x})$, symmetric stress–temperature tensor $\mathbf{M}(\mathbf{x})$, fixed reference temperature $\theta_0(\mathbf{x})$, body force $\mathbf{b}(\mathbf{x}, \tau)$, heat supply $r(\mathbf{x}, \tau)$, convection coefficient $h(\mathbf{x})$, and sink temperature $\vartheta_\infty(\mathbf{x}, \tau)$ are all explicit design fields.

The absolute temperature $\theta(\mathbf{x}, \tau) \equiv \vartheta + \theta_0$ is often used to characterize the thermal response in eqn (1); but as seen in the following, it is more convenient to use ϑ. τ is the independent time variable; t is the terminal time in the analysis interval $[0, t]$; and $\mathbf{x}$ denotes

† Subscripts represent components in a Cartesian coordinate system and the summation convention is used.

the position vector. All quantities are defined in the body B or its bounding surface ∂B (with outward unit normal vector $\mathbf{n}$) and are assumed smooth enough to justify the operations performed. dv and da represent differential elements in B and ∂B. Here, G is assumed to be differentiable with respect to the design; in practice this assumption is not always met (Haug *et al.*, 1986). Although G is defined in integral form, localized performance criteria over the spatial and/or time domains can be obtained by incorporating an appropriate weighting function.

The response of the system is implicity governed by the design and the mixed boundary-initial-value problem of thermoelasticity. The standard forms of the equation of motion and the energy balance equation are replaced in the following by the convolution expressions, (2) and (3), following a development by Carlson (1972). The nonstandard forms are introduced because they lead to a reciprocal theorem that is used to define the adjoint system in Section 3. If ϕ and ψ are scalar functions defined on $B\times[0,t]$, then the convolution operator is defined as $\phi*\psi(x,\tau)=\int_0^\tau \phi(x,\tau-\bar{\tau})\psi(x,\bar{\tau})\,d\bar{\tau}$. The convolution has the following properties (see Carlson, 1972, for a more detailed discussion): (i) $\phi*\psi=\psi*\phi$; (ii) $(\phi*\psi)*\omega=\phi*(\psi*\omega)=\phi*\psi*\omega$; (iii) $\phi*(\psi+\omega)=\phi\,\psi+\phi*\omega$; (iv) if ϕ is smooth in time, then $\phi*\dot{\psi}=\dot{\phi}*\psi+\phi(\cdot,0)\psi$. We define the generalized time and unit functions as $i(x,\tau)\equiv\tau$; and $1(x,\tau)\equiv 1$. Then the thermoelastic system equations are (Carlson, 1972):

$$i*S_{ij,j}+\mathscr{B}_i=\rho u_i \quad \text{in} \quad B\times[0,t] \tag{2}$$

$$-1*q_{i,i}+\mathscr{H}+\theta_0 M_{ij}E_{ij}=c\vartheta \quad \text{in} \quad B\times[0,t] \tag{3}$$

$$S_{ij}=C_{ijkl}E_{kl}+\vartheta M_{ij} \quad \text{in} \quad B\times[0,t] \tag{4}$$

$$q_i=-K_{ij}g_j \quad \text{in} \quad B\times[0,t] \tag{5}$$

$$E_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i}) \quad \text{in} \quad B\times[0,t] \tag{6}$$

$$g_i=\vartheta_{,i} \quad \text{in} \quad B\times[0,t] \tag{7}$$

$$s_i=S_{ij}n_j \quad \text{on} \quad \partial B\times[0,t] \tag{8}$$

$$q^s=q_i n_i \quad \text{on} \quad \partial B\times[0,t] \tag{9}$$

$$u_i=u_i^p \quad \text{on} \quad A_u\times[0,t] \tag{10}$$

$$s_i=s_i^p \quad \text{on} \quad A_s\times[0,t] \tag{11}$$

$$\vartheta=\vartheta^p \quad \text{on} \quad A_\vartheta\times[0,t] \tag{12}$$

$$q^s=q^p \quad \text{on} \quad A_q\times[0,t] \tag{13}$$

$$q^s=h(\vartheta-\vartheta_c) \quad \text{on} \quad A_C\times[0,t]. \tag{14}$$

In the above $(\ )_{,i}\equiv\partial(\ )/\partial x_i$; and symmetry of the stress tensor is assumed. The *pseudo body force* $\mathscr{B}(\mathbf{x},\tau)$, and *pseudo heat supply* $\mathscr{H}(\mathbf{x},\tau)$, are defined respectively by:

$$\mathscr{B}_i=i*b_i+\rho(u_i^0+\tau v_i^0) \tag{15}$$

$$\mathscr{H}=1*r+c\vartheta^0-\theta_0 M_{ij}E_{ij}^0. \tag{16}$$

The functions $\mathbf{u}^0(\mathbf{x})$, $\mathbf{v}^0(\mathbf{x})$, $\vartheta^0(\mathbf{x})$, and $\mathbf{E}^0(\mathbf{x})$ represent the initial displacement, velocity, temperature difference, and strain fields, respectively. These quantities are treated as design fields since they are specified explicitly in the analysis. A_u and A_s are complementary subsurfaces of ∂B as are A_ϑ, A_q, and A_C. A_u and A_s correspond to surfaces with prescribed displacement $\mathbf{u}^p(\mathbf{x}, \tau)$, traction $\mathbf{s}^p(\mathbf{x}, \tau)$: A_ϑ, A_q, and A_C have prescribed temperature difference $\vartheta^p(\mathbf{x}, \tau)$, surface flux $q^p(\mathbf{x}, \tau)$ and convection conditions. The relationship between the locations of surfaces A_u and A_s to surfaces A_ϑ, A_q, and A_C is arbitrary. Since $\mathbf{u}^p$, $\mathbf{s}^p$, ϑ^p and q^p are prescribed, they are also treated as design quantities.

We now show that eqn (2) must hold for a dynamic system governed by the equations of motion with initial displacements, $\mathbf{u}^0(\mathbf{x})$, and velocities, $\mathbf{v}^0(\mathbf{x})$. We take the convolution of the standard motion equation, $S_{ij,j}+b_i = \ddot{u}_i$, and then integrate the right-hand side by parts: $i * (S_{ij,j}+b_i) = i * (\rho \ddot{u}_i) = \rho \int_0^t (t-\tau)\ddot{u}(\tau)\,d\tau = \rho u_i - \rho[tv_i^0 - u_i^0]$. Combining this result with eqn (15) we recover eqn (2). We can demonstrate the converse result, that if eqn (2) is satisfied then the standard form of the equation of motion must be satisfied, by differentiating eqn (2) twice with respect to τ. Convolution property (iv) is used in the first differentiation. Thus, we can state that the equation of motion is satisfied if and only if eqn (2) is satisfied. Similar arguments can be used to relate eqn (3) to the standard form of the energy balance equation (in this case we only differentiate once with respect to τ).

The objective of a sensitivity analysis is to derive a relation for δG in which only explicit variations of the design fields are present. In Section 3, the material properties ($\mathbf{C}$, ρ, $\mathbf{K}$, c, $\mathbf{M}$, θ_0, and h) and load data ($\mathbf{b}$, r, $\mathbf{u}^p$, $\mathbf{s}^p$, ϑ^p, q^p, $\vartheta_\infty \mathbf{u}^0$, $\mathbf{v}^0$, $\mathbf{E}^0$, and ϑ^0) are varied with the shape held fixed; and an explicit expression δG_D is derived. In Section 4, the material properties and load data are fixed while the shape is varied; and an explicit shape sensitivity expression δG_X, is derived. The total variation δG is evaluated from the sum $\delta G_D + \delta G_X$.

3. VARIATIONS OF MATERIAL PROPERTIES AND LOAD DATA

The direct expression for δG_D is

$$\delta G_D = \int_0^t \Bigg[\int_B (f_{,C_{ijkl}} \delta C_{ijkl} + f_{,\rho} \delta \rho ^4 f_{,K_{ij}} \delta K_{ij} + f_{,c} \delta c + f_{,M_{ij}} \delta M_{ij} + f_{,C_{ijkl}} \delta C_{ijkl} + f_{,\rho}\, \delta \rho$$
$$+ f_{,\theta_0} \delta \theta_0 + f_{,b_i} \delta b_i + f_{,r} \delta r)\, dv + \int_{A_u} g_{,u_i^p} \delta u_i^p \, da + \int_{A_s} g_{,s_i^p} \delta s_i^p \, da$$
$$+ \int_{A_\vartheta} g_{,\vartheta^p} \delta \vartheta^p \, da + \int_{A_q} g_{,q^p} \delta q^p \, da + \int_{A_C} (g_{,h} \delta h + g_{,\vartheta_\infty} \delta \vartheta_\infty)\, da$$
$$+ \int_B (f_{,u_i} \delta u_i + f_{,E_{ij}} \delta E_{ij} + f_{,S_{ij}} \delta S_{ij} + f_{,\vartheta} \delta \vartheta + f_{,g_i} \delta g_i + f_{,q_i} \delta q_i)\, dv$$
$$+ \int_{A_u} g_{,s_i} \delta s_i \, da + \int_{A_s} g_{,u_i} \delta u_i \, da + \int_{A_\vartheta} g_{,q^s} \delta q^s \, da$$
$$+ \int_{A_q} g_{,\vartheta} \delta \vartheta \, da + \int_{A_C} g_{,\vartheta} \, da \Bigg] d\tau \tag{17}$$

where $a_{,b} \equiv \partial a / \partial b$.

The evaluation of this expression is not straightforward due to the presence of the implicit response variations ($\delta\mathbf{u}$, $\delta\vartheta$, $\delta\mathbf{E}$, $\delta\mathbf{g}$, $\delta\mathbf{S}$, $\delta\mathbf{q}$, δs, and δq^s) which must satisfy eqns (2)–(16) for the varied design. That is, the following equations must hold:

$$i * \delta S_{ij,j} + \delta \mathscr{B}_i = \delta\rho u_i + \rho \delta u_i \quad \text{in} \quad B \times [0, t] \tag{18}$$

$$-1 * \delta q_{i,i} + \delta \mathscr{R} + \delta\theta_0 M_{ij} E_{ij} + \theta_0 \delta M_{ij} E_{ij} + \theta_0 M_{ij} \delta E_{ij} = \delta c \vartheta + c \delta\vartheta \quad \text{in} \quad B \times [0, t] \tag{19}$$

$$\delta S_{ij} = \delta C_{ijkl} E_{kl} + C_{ijkl} \delta E_{kl} + \delta\vartheta M_{ij} + \vartheta \delta M_{ij} \quad \text{in} \quad B \times [0, t] \tag{20}$$

$$\delta q_i = -\delta K_{ij} g_j - K_{ij} \delta g_j \quad \text{in} \quad B \times [0, t] \tag{21}$$

$$\delta E_{ij} = \tfrac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \quad \text{in} \quad B \times [0, t] \tag{22}$$

$$\delta g_i = \delta \vartheta_{,i} \quad \text{in} \quad B \times [0, t] \tag{23}$$

$$\delta s_i = \delta S_{ij} n_j \quad \text{on} \quad \partial B \times [0, t] \tag{24}$$

$$\delta q^s = \delta q_i n_i \quad \text{on} \quad \partial B \times [0, t] \tag{25}$$

$$\delta u_i = \delta u_i^p \quad \text{on} \quad A_u \times [0, t] \tag{26}$$

$$\delta s_i = \delta s_i^p \quad \text{on} \quad A_s \times [0, t] \tag{27}$$

$$\delta \vartheta = \delta \vartheta^p \quad \text{on} \quad A_\vartheta \times [0, t] \tag{28}$$

$$\delta q^s = \delta q_p \quad \text{on} \quad A_q \times [0, t] \tag{29}$$

$$\delta q^s = \delta h(\vartheta - \vartheta_x) + h(\delta \vartheta - \delta \vartheta_x) \quad \text{on} \quad A_C \times [0, t] \tag{30}$$

where

$$\delta \mathscr{B}_i = i * \delta b_i + \delta \rho (u_i^0 + t v_i^0) + \rho (\delta u_i^0 + t \delta v_i^0) \tag{31}$$

$$\delta \mathscr{R} = 1 * \delta r + \delta c \vartheta^0 + c \delta \vartheta^0 - \delta \theta_0 M_{ij} E_{ij}^0 - \theta_0 \delta M_{ij} E_{ij}^0 - \theta_0 M_{ij} \delta E_{ij}^0. \tag{32}$$

In sensitivity analysis of elliptic problems, such as static elasticity, reciprocal theorems or mutual energy principles that result from the self-adjoint nature of the governing equations can be invoked to eliminate the implicit response variations (Phelan and Haber, 1989). In these methods, the reciprocal theorem relates the load and response fields of a fictitious adjoint system to the design and response variations of the real system. Unfortunately, the standard forms of the equation of motion and the energy balance equations for the transient thermoelastic problem are not self-adjoint; so they do not lead directly to weak-form reciprocal relations. Now the utility of the nonstandard forms of eqns (2) and (3) becomes apparent: despite the lack of self-adjoint operators in the standard equations, a relationship of reciprocal form can be generated (Carlson, 1972) because of the symmetry of the convolution operator; and we can use the reciprocal relation to eliminate the implicit design variations in the sensitivity expression. Further, the initial conditions are now incorporated in the formulation, and their variations can be considered in the sensitivity formulation.

It is not necessary to use the convolution equations in solving the real and adjoint systems—their key function here is to facilitate the formulation of the adjoint system and the definition of the adjoint data. In fact, we have used the standard equations and conventional time integration methods in formulating our numerical solution algorithms for the real and adjoint problems (see Section 6).

The adjoint system equations are presented next. A tilde denotes an adjoint load or response quantity—the only fields that differ between the real and adjoint systems. Note that the adjoint system is defined on the same space-time domain as the real system and that the two systems share the same material properties. In fact, the adjoint and the real system equations are of nearly the same form, except for the introduction of applied stress, strain, temperature gradient and heat flux fields in the adjoint equations (to obtain identical system equations, the applied terms could be formally introduced to the real system equations and set to zero). A reciprocal theorem is introduced below and a method is presented to determine the adjoint load data needed to relate the implicit response variations of the real system to the response of the adjoint system. The adjoint system equations are:

$$i * \tilde{S}_{ij,j} + \tilde{\mathscr{B}}_i = \rho \tilde{u}_i \quad \text{in} \quad B \times [0, t] \tag{33}$$

$$-1 * \tilde{q}_{i,i} + \tilde{\mathscr{R}} + \theta_0 M_{ij} \dot{\tilde{E}}_{ij} = c\tilde{\vartheta} \quad \text{in} \quad B \times [0, t] \tag{34}$$

$$\tilde{S}_{ij} = C_{ijkl}(\tilde{E}_{kl} - \tilde{E}^A_{kl}) + \tilde{S}^A_{ij} + \tilde{\vartheta} M_{ij} \quad \text{in} \quad B \times [0, t] \tag{35}$$

$$\tilde{q}_i = -K_{ij}(\tilde{g}_j - \tilde{g}^A_j) + \tilde{q}^A_i \quad \text{in} \quad B \times [0, t] \tag{36}$$

$$\tilde{E}_{ij} = \tfrac{1}{2}(\tilde{u}_{i,j} + \tilde{u}_{j,i}) \quad \text{in} \quad B \times [0, t] \tag{37}$$

$$\tilde{g}_i = \tilde{\vartheta}_{,i} \quad \text{in} \quad B \times [0, t] \tag{38}$$

$$\tilde{s}_i = \tilde{S}_{ij} n_j \quad \text{on} \quad \partial B \times [0, t] \tag{39}$$

$$\tilde{q}^s = \tilde{q}_i n_i \quad \text{on} \quad \partial B \times [0, t] \tag{40}$$

$$\tilde{u}_i = \tilde{u}^p_i \quad \text{on} \quad A_u \times [0, t] \tag{41}$$

$$\tilde{s}_i = \tilde{s}^p_i \quad \text{on} \quad A_s \times [0, t] \tag{42}$$

$$\tilde{\vartheta} = \tilde{\vartheta}^p \quad \text{on} \quad A_\vartheta \times [0, t] \tag{43}$$

$$\tilde{q}^s = \tilde{q}_p \quad \text{on} \quad A_q \times [0, t] \tag{44}$$

$$\tilde{q}^s = h(\tilde{\vartheta} - \tilde{\theta}_\infty) \quad \text{on} \quad A_C \times [0, t] \tag{45}$$

where the adjoint load data consist of the pseudo body force $\tilde{\mathscr{B}}(\mathbf{x}, \tau)$, pseudo heat supply $\tilde{\mathscr{R}}(\mathbf{x}, \tau)$, applied stress tensor $\tilde{\mathbf{S}}^A(\mathbf{x}, \tau)$, applied strain tensor $\tilde{\mathbf{E}}^A(\mathbf{x}, \tau)$, applied temperature gradient $\tilde{\mathbf{g}}^A(\mathbf{x}, \tau)$, applied heat flux vector $\tilde{\mathbf{q}}^A(\mathbf{x}, \tau)$, prescribed displacement $\tilde{\mathbf{u}}^p(\mathbf{x}, \tau)$, prescribed surface traction $\tilde{\mathbf{s}}^p(\mathbf{x}, \tau)$, prescribed relative temperature $\tilde{\vartheta}^p(\mathbf{x}, \tau)$, prescribed surface flux $\tilde{q}^p(\mathbf{x}, \tau)$, and sink temperature $\tilde{\theta}_\infty(\mathbf{x}, \tau)$. The applied terms are, in some ways, analogous to initial stress or strain terms common to elasticity problems. However, the applied terms are time-dependent and need not be symmetric. These data are chosen to annihilate the integrands in eqn (17) with implicit response variations, as explained below.

Carlson (1972) presented a reciprocal theorem valid for dynamic thermoelastic systems. In the present formulation, this theorem is modified to relate variations in the real system [eqns (18)–(32)] to their counterparts in the adjoint system [eqns (33)–(45)].

Theorem 1. *Suppose a thermoelastic system is subjected to two separate dynamic load systems, the real and adjoint. Next, suppose the material properties and load data of the real system are varied,* $\mathbf{C} \to \mathbf{C} + \delta\mathbf{C}$, $\rho \to \rho + \delta\rho$, $\mathbf{K} \to \mathbf{K} + \delta\mathbf{K}$, $c \to c + \delta c$, $\mathbf{M} \to \mathbf{M} + \delta\mathbf{M}$, $\theta_0 \to \theta_0 + \delta\theta_0$, $\mathbf{b} \to \mathbf{b} + \delta\mathbf{b}$, $r \to r + \delta r$, $\mathbf{u}^p \to \mathbf{u}^p + \delta\mathbf{u}^p$, $\mathbf{s}^p \to \mathbf{s}^p + \delta\mathbf{s}^p$, $\vartheta^p \to \vartheta^p + \delta\vartheta^p$, $q^p \to q^p + \delta p^p$, $h \to h + \delta h$, $\vartheta_\infty \to \vartheta_\infty + \delta\vartheta_\infty$, $\mathbf{u}^0 \to \mathbf{u}^0 + \delta\mathbf{u}^0$, $\mathbf{v}^0 \to \mathbf{v}^0 + \delta\mathbf{v}^0$, $\vartheta^0 \to \vartheta^0 + \delta\vartheta^0$, *and* $\mathbf{E}^0 \to \mathbf{E}^0 + \delta\mathbf{E}^0$. *Then*

$$i * \int_B \tilde{E}_{ij} * C_{ijkl}\delta E_{kl}\,\mathrm{d}v - \frac{i}{\theta_0} * 1 * \int_B \tilde{g}_i * K_{ij}\delta g_j\,\mathrm{d}v$$
$$= i * \int_B \delta E_{ij} * C_{ijkl}\tilde{E}_{kl}\,\mathrm{d}v - \frac{i}{\theta_0} * 1 * \int_B \delta g_i * K_{ij}\tilde{g}_j\,dv \tag{46}$$

and

$$i * \int_{\partial B} \tilde{u}_i * \delta s_i \, da + \int_B \tilde{u}_i * (\delta \mathscr{B}_i - \delta \rho \ddot{u}_i) \, dv - i * \int_B \tilde{E}_{ij} * (\delta C_{ijkl} E_{kl} + \vartheta \delta M_{ij}) \, dv$$

$$+ \frac{i}{\theta_0} * 1 * \int_{\partial B} \tilde{\theta} * \delta q^s \, da + \frac{i}{\theta_0} * \int_B \tilde{\vartheta} * (\delta c \vartheta - \delta \mathscr{R} - \delta \theta_0 M_{ij} E_{ij}) \, dv$$

$$+ \frac{i}{\theta_0} * 1 * \int_B \tilde{g}_i * \delta K_{ij} g_j \, dv = i * \int_{\partial B} \tilde{s}_i * \delta u_i \, da + \int_B \tilde{\mathscr{B}}_i * \delta u_i \, dv$$

$$+ i * \int_B (\tilde{E}^A_{ij} C_{ijkl} - \tilde{S}^A_{kl}) * \delta E_{kl} \, dv + \frac{i}{\theta_0} * 1 * \int_{\partial B} \tilde{q}^s * \delta \vartheta \, da - \frac{i}{\theta_0} * \int_B \tilde{\mathscr{R}} * \delta \vartheta \, dv$$

$$- \frac{i}{\theta_0} * 1 * \int_B (K_{ij} \tilde{g}^A_j + \tilde{q}^A_i) * \delta g_i \, dv. \tag{47}$$

The symmetry of **C**, **K**, and the convolution verify the first equality. Several applications of the divergence theorem, symmetry of **C**, **K**, **M**, and the convolution, and eqns (18)–(45) transform the first equality to the second.

This theorem is now used to eliminate the integrands which contain implicit variations in δG_p. First, the implicit and explicit variations in eqn (47) are isolated;

$$i * \int_B \tilde{E}^A_{ij} \delta C_{ijkl} * E_{kl} \, dv - \frac{i}{\theta_0} * 1 * \int_B g_i \delta K_{ij} * \tilde{g}^A_j \, dv$$

is added to each side; and eqns (18)–(45) are substituted to give

$$- i * \int_{A_u} \tilde{s}_i * \delta u^p_i \, da + i * \int_{A_s} \tilde{u}_i * \delta s^p_i \, da + \int_B \tilde{u}_i * (\delta \mathscr{B}_i - \delta \rho \ddot{u}_i) \, dv - i * \int_B (\tilde{E}_{ij} * \vartheta \delta M_{ij}$$

$$+ (\tilde{E}_{ij} - \tilde{E}^A_{ij}) * \delta C_{ijkl} E_{kl}) \, dv - \frac{i}{\theta_0} * 1 * \int_{A_\vartheta} \tilde{q}^s * \delta \vartheta^p \, da + \frac{i}{\theta_0} * 1 * \int_{A_q} \tilde{\vartheta} * \delta q^p \, da$$

$$+ \frac{i}{\theta_0} * 1 * \int_{A_c} \tilde{\vartheta} * (\delta h (\vartheta - \vartheta_c) - h \delta \vartheta_c) \, da + \frac{i}{\theta_0} * \int_B \tilde{\vartheta} * (\delta c \vartheta - \delta \mathscr{R} - \delta \theta_0 M_{ij} E_{ij}$$

$$- \theta_0 \delta M_{ij} E_{ij}) \, dv + \frac{i}{\theta_0} * 1 * \int_B (\tilde{g}_i - \tilde{g}^A_i) * \delta K_{ij} g_j \, dv = - i * \int_{A_u} \tilde{u}^p_i * \delta s_i \, da$$

$$+ i * \int_{A_s} \tilde{s}^p_i * \delta u_i \, da + \int_B \tilde{\mathscr{B}}_i * \delta u_i \, dv + i * \int_B (\tilde{E}^A_{ij} * \delta S_{ij} - \tilde{S}^A_{ij} * \delta E_{ij}) \, dv$$

$$- \frac{i}{\theta_0} * 1 * \int_{A_\vartheta} \tilde{\vartheta}^p * \delta q^s \, da + \frac{i}{\theta_0} * 1 * \int_{A_q} \tilde{q}^p * \delta \vartheta \, da - \frac{i}{\theta_0} * 1 * \int_{A_c} \tilde{\vartheta} * h \delta \vartheta \, da$$

$$- \frac{i}{\theta_0} * \int_B \tilde{\mathscr{R}} * \delta \vartheta \, dv - \frac{i}{\theta_0} * 1 * \int_B (- \tilde{g}^A_i * \delta q_i + \tilde{q}^A_i * \delta g_i) \, dv. \tag{48}$$

Equation (48) is differentiated twice with respect to time to relate the right-hand side of this equation to the integrands that contain response variations in δG_p:

$$-\int_{A_u} \tilde{s}_i * \delta u_i^p \,da + \int_{A_s} \tilde{u}_i * \delta s_i^p \,da + \int_B (\tilde{u}_i * (\delta b_i - \delta\rho\ddot{u}_i) + \dot{\tilde{u}}_i|_{(\mathbf{x},t)}\rho\delta u_i^0 + \tilde{u}_i|_{(\mathbf{x},t)}\rho\delta v_i^0)\,dv$$

$$-\int_B (\tilde{E}_{ij} * \vartheta\delta M_{ij} + (\tilde{E}_{ij} - \tilde{E}_{ij}^A) * \delta C_{ijkl}E_{kl})\,dv - \frac{1}{\theta_0} * \int_{A_\vartheta} \tilde{q}^s * \delta\vartheta^p \,da + \frac{1}{\theta_0} * \int_{A_q} \tilde{\vartheta} * \delta q^p \,da$$

$$+ \frac{1}{\theta_0} * \int_{A_C} \tilde{\vartheta} * (\delta h(\vartheta - \vartheta_\infty) - h\delta\vartheta_\infty)\,da + \frac{1}{\theta_0}\int_B \tilde{\vartheta} * (\delta c\vartheta - \delta\mathscr{R} - \delta\theta_0 M_{ij}E_{ij}$$

$$-\theta_0\delta M_{ij}E_{ij})\,dv + \frac{1}{\theta_0} * \int_B (\tilde{g}_i - \tilde{g}_i^A) * \delta K_{ij}\,g_j\,dv = -\int_{A_u} \tilde{u}_i^p * \delta s_i\,da + \int_{A_s} \tilde{s}_i^p * \delta u_i\,da$$

$$+\int_B (\tilde{b}_i * \delta u_i + \tilde{u}_i^0\rho\delta\dot{u}_i(t + \tilde{v}_i^0\rho\delta u_i|_{(\mathbf{x},t)}))\,dv + \int_B (\tilde{E}_{ij}^A * \delta S_{ij} - \tilde{S}_{ij}^A * \delta E_{ij})\,dv$$

$$-\frac{1}{\theta_0} * \int_{A_\vartheta} \tilde{\vartheta}^p * \delta q^s\,da + \frac{1}{\theta_0} * \int_{A_q} \tilde{q}^p * \delta\vartheta\,da - \frac{1}{\theta_0} * \int_{A_C} \tilde{\theta} * h\delta\vartheta\,da - \frac{1}{\theta_0}\int_B \tilde{\mathscr{R}} * \delta\vartheta\,dv$$

$$-\frac{1}{\theta_0} * \int_B (-\tilde{g}_i^A * \delta q_i + \tilde{q}_i^A * \delta g_i)\,dv \tag{49}$$

where $(\dot{\ }) \equiv d(\ )/dt$ and $(\ddot{\ }) \equiv d^2(\ )/dt^2$.

The right-hand side of this equation is equated to the integrands which contain implicit variations in δG_p, to define the adjoint data as:

$$\tilde{b}_i(\mathbf{x}, t-\tau) = f_{,u_i}|_{(\mathbf{x},\tau)} \quad \text{in} \quad B \times [0, t] \tag{50}$$

$$\tilde{S}_{ij}^A(\mathbf{x}, t-\tau) = -f_{,E_{ij}}|_{(\mathbf{x},\tau)} \quad \text{in} \quad B \times [0, t] \tag{51}$$

$$\tilde{E}_{ij}^A(\mathbf{x}, t-\tau) = f_{,S_{ij}}|_{(\mathbf{x},\tau)} \quad \text{in} \quad B \times [0, t] \tag{52}$$

$$\tilde{\mathscr{R}}(\mathbf{x}, t-\tau) = -\theta_0 f_{,\vartheta}|_{(\mathbf{x},\tau)} \quad \text{in} \quad B \times [0, t] \tag{53}$$

$$1 * \tilde{q}_i^A(\mathbf{x}, t-\tau) = -\theta_0 f_{,g_i}|_{(\mathbf{x},\tau)} \quad \text{in} \quad B \times [0, t] \tag{54}$$

$$1 * \tilde{g}_i^A(\mathbf{x}, t-\tau) = \theta_0 f_{,q_i}|_{(\mathbf{x},\tau)} \quad \text{in} \quad B \times [0, t] \tag{55}$$

$$\tilde{u}^p(\mathbf{x}, t-\tau) = -g_{,s_i}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_u \times [0, t] \tag{56}$$

$$\tilde{s}^p(\mathbf{x}, t-\tau) = g_{,u_i}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_s \times [0, t] \tag{57}$$

$$1 * \tilde{\vartheta}^p(\mathbf{x}, t-\tau) = -\theta_0 g_{,q^s}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_\vartheta \times [0, t] \tag{58}$$

$$1 * \tilde{q}^p(\mathbf{x}, t-\tau) = \theta_0 g_{,\vartheta}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_q \times [0, t] \tag{59}$$

$$1 * \tilde{\vartheta}_\infty(\mathbf{x}, t-\tau) = -\frac{\theta_0}{h} g_{,\vartheta}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_C \times [0, t] \tag{60}$$

$$\tilde{u}_i^0 = 0 \quad \text{in} \quad B \tag{61}$$

$$\tilde{v}_i^0 = 0 \quad \text{in} \quad B. \tag{62}$$

The left-hand side of eqn (49) replaces the integrands in δG_p which contain variations of the response fields and the explicit sensitivity is obtained:

$$
\begin{aligned}
\delta G_D = \int_0^t \Bigg(\int_B & (f_{,C_{ijkl}} \delta C_{ijkl} + f_{,\rho}\delta\rho + f_{,K_{ij}}\delta K_{ij} + f_{,c}\delta c + f_{,M_{ij}}\delta M_{ij} \\
& + f_{,\theta_0}\delta\theta_0 + f_{,b_i}\delta b_i + f_{,r}\delta r)\,\mathrm{d}v + \int_{A_u} g_{,u_i^p}\delta u_i^p\,\mathrm{d}a + \int_{A_s} g_{,s_i^p}\delta s_i^p\,\mathrm{d}a \\
& + \int_{A_\vartheta} g_{,\vartheta^p}\delta\vartheta^p\,\mathrm{d}a + \int_{A_q} g_{,q^p}\delta q^p\,\mathrm{d}a + \int_{A_C} (g_{,h}\delta h + g_{,\vartheta_x}\delta\vartheta_x)\,\mathrm{d}a \Bigg)\mathrm{d}\tau \\
& - \int_{A_u} \tilde{s}_i * \delta u_i^p\,\mathrm{d}a + \int_{A_s} \tilde{u}_i * \delta s_i^p\,\mathrm{d}a + \int_B (\tilde{u}_i * (\delta b_i - \delta\rho\ddot{u}_i) \\
& + \dot{\tilde{u}}_i|_{(\mathbf{x},t)}\rho\delta u_i^0 + \tilde{u}_i|_{(\mathbf{x},t)}\rho\delta v_i^0)\,\mathrm{d}v - \int_B (\tilde{E}_{ij} * \vartheta\delta M_{ij} \\
& + (\tilde{E}_{ij} - \tilde{E}_{ij}^A) * \delta C_{ijkl}E_{kl})\,\mathrm{d}v - \frac{1}{\theta_0} * \int_{A_\vartheta} \tilde{q}^s * \delta\vartheta^p\,\mathrm{d}a + \frac{1}{\theta_0} * \int_{A_q} \tilde{\vartheta} * \delta q^p\,\mathrm{d}a \\
& + \frac{1}{\theta_0} * \int_{A_C} \tilde{\vartheta} * (\delta h(\vartheta - \vartheta_x) - h\delta\vartheta_x)\,\mathrm{d}a + \frac{1}{\theta_0}\int_B \tilde{\vartheta} * (\delta c\vartheta - \delta\mathscr{R} \\
& - \delta\theta_0 M_{ij}E_{ij} - \theta_0\delta M_{ij}E_{ij})\,\mathrm{d}v + \frac{1}{\theta_0} * \int_B (\tilde{g}_i - \tilde{g}_i^A) * \delta K_{ij} g_j\,\mathrm{d}v.
\end{aligned} \tag{63}
$$

Note that the solution to the adjoint problem must be available to render eqn (63) explicit. If the finite element method is used to evaluate the real response, then the adjoint response can be computed in an efficient manner. At each time step, the adjoint load vector, defined by eqns (50)–(62) is assembled and back-substituted into the decomposed stiffness matrix that was used for the real analysis. If uniform time steps are used for the real analysis, then only one stiffness matrix decomposition need be performed to evaluate both the real and adjoint responses (Tortorelli and Haber, 1989).

4. VARIATIONS OF SHAPE

To formulate explicit shape sensitivities domain parameterization is used (Haber, 1987; Phelan and Haber, 1989). A reference configuration B^r, described in an independent reference coordinate system with position vector $\mathbf{r}$, is introduced such that

$$\mathbf{x}(\mathbf{r}) : B^r \rightarrow B \tag{64}$$

where $\mathbf{x}$ is a deformation-like mapping. The configuration B is defined by the image $\mathbf{x}(B^r)$ and the variants of this configuration by $\mathbf{x}(B^r) + \delta\mathbf{x}(B^r)$. This method for obtaining shape sensitivities was also proposed by Cea (1981a,b). If the isoparametric finite element method is used to perform the real analysis, then the isoparametric mapping is a natural choice to locally define $\mathbf{x}$. Over each element, $\mathbf{x} = \sum_{\alpha=1}^{N} \chi_\alpha H_\alpha$ where χ_α is the coordinate vector of the αth node in the element, H_α is the corresponding shape function, and N is the number of nodes in the element. With this mapping, the sensitivities are expressed with respect to variations of the node coordinates which serve as the fundamental design parameters. Typically, the coordinates are linked to a smaller number of global geometric design parameters (e.g. Braibant and Fleury, 1984).

The next step in the derivation is to rewrite all the field quantities as functions of $\mathbf{r}$ over the reference domain. For example, $\mathbf{b} = \mathbf{b}(\mathbf{r}, \tau)$ on $B^r \times [0, t]$. G is transformed to the reference configuration by the change of variable theorem (Hildebrand, 1976):

$$G = \int_0^t \left[\int_{B^r} f(u_i, E_{ij}, S_{ij}, \vartheta, g_i, q_i, C_{ijkl}, \rho, K_{ij}, c, M_{ij}, \theta_0, b_i, r) J \, dv^r + \int_{\partial B^r} g(u_i, s_i, \vartheta, q^s, h, \vartheta_\infty) K \, da^r \right] d\tau \quad (65)$$

where $J \equiv dv/dv^r$ is the determinant of the Jacobian tensor $\mathbf{J}$ with components $J_{ij} \equiv x_{i,j}$; $(\ )_{,j} \equiv \partial(\ )/\partial \tau_j$; and $K \equiv da/da^r$ is a surface area metric (Haug *et al.*, 1986; Tortorelli *et al.*, 1990). dv^r and da^r are differential elements of B^r and ∂B^r.

A variation $\delta \mathbf{x}$ gives

$$\delta G_x = \int_0^t \left[\int_{B^r} (f_{,u_i}\delta u_i + f_{,E_{ij}}\delta E_{ij} + f_{,S_{ij}}\delta S_{ij} + f_{,\vartheta}\delta\vartheta + f_{,g_i}\delta g_i + f_{,q_i}\delta q_i) J \, dv^r + \int_{A_u^r} g_{,s_i}\delta s_i K \, da^r + \int_{A_s^r} g_{,u_i}\delta u_i K \, da^r + \int_{A_\vartheta^r} g_{,q^s}\delta q^s K \, da^r + \int_{A_q^r} g_{,\vartheta}\delta\vartheta K \, da^r + \int_{A_c^r} g_{,\vartheta}\delta\vartheta K \, da^r + \int_{B^r} f \delta J \, dv^r + \int_{\partial B^r} g \delta K \, da^r \right] d\tau \quad (66)$$

where δJ and δK are explicit functions of $\mathbf{x}$ and its variation $\delta \mathbf{x}$ (Tortorelli *et al.*, 1990).

Equations (2)–(14) must also be transformed to the reference domain. This is accomplished by expressing the conservation laws over B^r:

$$\int_{B^r} \mathscr{B}_i(\mathbf{r}, \tau) J \, dv^r + i * \int_{\partial B^r} s_i(\mathbf{r}, \tau) K \, da^r = \int_{B^r} \rho u_i(\mathbf{r}, \tau) J \, dv^r \quad \text{on} \quad [0, t] \quad (67)$$

$$\int_{B^r} (\mathscr{R}(\mathbf{r}, \tau) + 1 * \theta_0 M_{ij}(\mathbf{r}, \tau) E_{ij}(\mathbf{r}, \tau)) J \, dv^r - 1 * \int_{\partial B^r} q^s(\mathbf{r}, \tau) K \, da^r = \int_{B^r} c\vartheta(\mathbf{r}, r) J \, dv^r \quad \text{on} \quad [0, t] \quad (68)$$

and utilizing the divergence theorem, Nanson's relation (Bathe, 1982), and the chain rule:

$$i * (J J_{jm}^{-1} S_{mi})_{,j} + \mathscr{B}_i J = \rho u_i J \quad \text{in} \quad B^r \times [0, t] \quad (69)$$

$$-1 * (q_i J J_{mi}^{-1})_{,m} + \mathscr{R} J + \theta_0 M_{ij} E_{ij} J = c\vartheta J \quad \text{in} \quad B^r \times [0, t] \quad (70)$$

$$S_{ij} = C_{ijkl} E_{kl} + \vartheta M_{ij} \quad \text{in} \quad B^r \times [0, t] \quad (71)$$

$$q_i = -K_{ij} g_j \quad \text{in} \quad B^r \times [0, t] \quad (72)$$

$$E_{ij} = \tfrac{1}{2}(u_{i,m} J_{mj}^{-1} + u_{j,m} J_{mi}^{-1}) \quad \text{in} \quad B^r \times [0, t] \quad (73)$$

$$g_i = \vartheta_{,m} J_{mi}^{-1} \quad \text{in} \quad B^r \times [0, t] \quad (74)$$

$$s_i K = J J_{jm}^{-1} S_{mi} n_j^r \quad \text{on} \quad \partial B^r \times [0, t] \quad (75)$$

$$q^s K = q_i J J_{mi}^{-1} n_m^r \quad \text{on} \quad \partial B^r \times [0, t] \quad (76)$$

$$u_i = u_i^p \quad \text{on} \quad A_u^r \times [0, t] \quad (77)$$

$$s_i = s_i^p \quad \text{on} \quad A_s^r \times [0, t] \quad (78)$$

$$\vartheta = \vartheta^p \quad \text{on} \quad A_\vartheta^r \times [0, t] \quad (79)$$

$$q^s = q^p \quad \text{on} \quad A_q^r \times [0, t] \tag{80}$$

$$q^s = h(\vartheta - \vartheta_\infty) \quad \text{on} \quad A_C^r \times [0, t]. \tag{81}$$

This transformation is similar to one presented in Tortorelli *et al.* (1990) for elastodynamic systems.

If the governing equations and boundary conditions are satisfied in the current design, then the variation of eqns (69)–(81) provides equations which ensure satisfaction of these relations in all neighboring designs. For the case of shape variations, this condition is expressed as

$$i * (\delta J J_{jm}^{-1} S_{mi})_{,j} + i * (J \delta J_{jm}^{-1} S_{mi})_{,j} + i * (J J_{jm}^{-1} \delta S_{mi})_{,j} + \mathscr{B}_i \delta J = \rho \delta u_i J + \rho u_i \delta J \quad \text{in} \quad B^r \times [0, t] \tag{82}$$

$$-1 * (\delta q_i J J_{mi}^{-1})_{,m} - 1 * (q_i \delta J J_{mi}^{-1})_{,m} - 1 * (q_i J \delta J_{mi}^{-1})_{,m} + \mathscr{R} \delta J + \theta_0 M_{ij} \delta E_{ij} J + \theta_0 M_{ij} E_{ij} \delta J = c \delta \vartheta J + c \vartheta \delta J \quad \text{in} \quad B^r \times [0, t] \tag{83}$$

$$\delta S_{ij} = C_{ijkl} \delta E_{kl} + \delta \vartheta M_{ij} \quad \text{in} \quad B^r \times [0, t] \tag{84}$$

$$\delta q_i = -K_{ij} \delta g_j \quad \text{in} \quad B^r \times [0, t] \tag{85}$$

$$\delta E_{ij} = \tfrac{1}{2}(\delta u_{i,m} J_{mj}^{-1} + \delta u_{j,m} J_{mi}^{-1}) + \tfrac{1}{2}(u_{i,m} \delta J_{mj}^{-1} + u_{j,m} \delta J_{mi}^{-1}) \quad \text{in} \quad B^r \times [0, t] \tag{86}$$

$$\delta g_i = \delta \vartheta_{,m} J_{mi}^{-1} + \vartheta_{,m} \delta J_{mi}^{-1} \quad \text{in} \quad B^r \times [0, t] \tag{87}$$

$$\delta s_i K + s_i \delta K = \delta J J_{jm}^{-1} S_{mi} n_j^r + J \delta J_{jm}^{-1} S_{mi} n_j^r + J J_{jm}^{-1} \delta S_{mi} n_j^r \quad \text{on} \quad \partial B^r \times [0, t] \tag{88}$$

$$\delta q^s K + q^s \delta K = \delta q_i J J_{mi}^{-1} n_m^r + q_i \delta J J_{mi}^{-1} n_m^r + q_i J \delta J_{mi}^{-1} n_m^r \quad \text{on} \quad \partial B^r \times [0, t] \tag{89}$$

$$\delta u_i = 0 \quad \text{on} \quad A_u^r \times [0, t] \tag{90}$$

$$\delta s_i = 0 \quad \text{on} \quad A_s^r \times [0, t] \tag{91}$$

$$\delta \vartheta = 0 \quad \text{on} \quad A_\vartheta^r \times [0, t] \tag{92}$$

$$\delta q^s = 0 \quad \text{on} \quad A_q^r \times [0, t] \tag{93}$$

$$\delta q^s = h \delta \vartheta \quad \text{on} \quad A_C^r \times [0, t] \tag{94}$$

where $\delta \mathbf{J}^{-1}$ is also an explicit function of the current shape and its variation (Tortorelli *et al.*, 1989, 1990).

A theorem for shape variations, based on reciprocity and similar to Theorem 1, is constructed from eqns (82)–(94).

Theorem 2. *Suppose a thermoelastic system is subjected to two separate dynamic load systems, the real and adjoint. Next, suppose the domain of the thermodynamic system is varied* $\mathbf{x} \to \mathbf{x} + \delta \mathbf{x}$, *then*

$$i * \int_{B^r} \tilde{E}_{ij} C_{ijkl} * \delta u_{k,m} J_{ml}^{-1} J \, dv^r - i * 1 * \int_{B^r} \tilde{g}_i K_{ij} * \delta \vartheta_{,m} J_{mj}^{-1} J \, dv^r = i * \int_{B^r} \delta u_{i,m} J_{mj}^{-1} C_{ijkl} * \tilde{E}_{kl} J \, dv^r - i * 1 * \int_{B^r} \delta \vartheta_{,m} J_{mi}^{-1} K_{ij} * \tilde{g}_j J \, dv^r \tag{95}$$

and

$$i * \int_{\partial B^r} (\delta s_i K + s_i \delta K) * \tilde{u}_i \, da^r - i * \int_{B^r} (\delta J J_{mj}^{-1} S_{ij} + J \delta J_{mj}^{-1} S_{ij}) * \tilde{u}_{i,m} \, dv^r$$

$$- \int_{B^r} (\rho u_i - \mathscr{B}_i) * \tilde{u}_i \delta J \, dv^r - i * \int_{B^r} u_{i,m} J \delta J_{mj}^{-1} C_{ijkl} * \tilde{E}_{kl} J \, dv^r$$

$$+ \frac{i}{\theta_0} * 1 * \int_{\partial B^r} (q^s \delta K + \delta q^s K) * \tilde{\vartheta} \, da^r - \frac{i}{\theta_0} * 1 * \int_{B^r} (q_i \delta J_{mi}^{-1} + q_i J \delta J_{mi}^{-1}) * \tilde{\theta}_{,m} \, dv^r$$

$$+ \frac{i}{\theta_0} * \int_{B^r} (c \vartheta \delta J - \theta_0 M_{ij} E_{ij} \delta J - \theta_0 M_{ij} u_{i,m} \delta J_{mj}^{-1} - \mathscr{R} \delta J) * \tilde{\vartheta} \, dv^r$$

$$+ \frac{i}{\theta_0} * 1 * \int_{B^r} \vartheta_{,m} \delta J_{mi}^{-1} K_{ij} * \tilde{g}_j J \, dv^r = i * \int_{\partial B^r} \tilde{s}_i * \delta u_i K \, da^r + \int_{B^r} \tilde{\mathscr{B}}_i * \delta u_i J \, dv^r$$

$$+ i * \int_{B^r} (C_{ijkl} \tilde{E}_{kl}^A - \tilde{S}_{ij}^A) * \delta u_{i,m} J \, dv^r + \frac{i}{\theta_0} * 1 * \int_{\partial B^r} \tilde{q}^s * \delta \vartheta K \, da^r$$

$$- \frac{i}{\theta_0} * \int_{B^r} \tilde{\mathscr{R}} * \delta \vartheta J \, dv^r - \frac{i}{\theta_0} * 1 * \int_{B^r} (K_{ij} \tilde{g}_j^A + \tilde{q}_i^A) * \delta \vartheta_{,m} J_{mi}^{-1} J \, dv^r. \tag{96}$$

Proof of this theorem is similar to the verification of Theorem 1, after the appropriate transformations to the reference configuration have been performed.

The manipulation of eqn (96) to isolate implicit and explicit variations, the addition of

$$i * \int_{B^r} u_{i,m} * (C_{ijkl} \tilde{E}_{kl}^A - \tilde{S}_{ij}^A) \delta J_{mj}^{-1} J \, dv^r - i * 1 * \int_{B^r} \vartheta_{,m} * (K_{ij} \tilde{g}_j^A + \tilde{q}_i^A) \delta J_{mi}^{-1} J \, dv^r$$

to each side, the substitution of eqns (33)–(45) and (82)–(94), and two time differentiations yields

$$\int_{\partial B^r} s_i * \tilde{u}_i \delta K \, da^r - \int_{B^r} (S_{ij} * \tilde{u}_{i,m} + \tilde{S}_{ij} * u_{i,m}) \delta J_{mj}^{-1} J \, dv^r - \int_{B^r} (S_{ij} * \tilde{E}_{ij}$$

$$+ (\rho \ddot{u}_i - b_i) * \tilde{u}_i) \delta J \, dv^r + \frac{1}{\theta_0} * \int_{\partial B^r} q^s * \tilde{\vartheta} \delta K \, da^r - \frac{1}{\theta_0} * \int_{B^r} (q_i * \tilde{\vartheta}_{,m} + \tilde{q}_i * \vartheta_{,m}) \delta J_{mi}^{-1} J \, dv^r$$

$$+ \frac{1}{\theta_0} * \int_{B^r} (-q_i * \tilde{g}_i + (c \vartheta - \theta_0 M_{ij} E_{ij} - \mathscr{R}) * \tilde{\vartheta}) \delta J \, dv^r = \int_0^t \left[- \int_{A_u^r} \tilde{u}_i^p * \delta s_i K \, da^r \right.$$

$$+ \int_{A_s^r} \tilde{s}_i^p * \delta u_i K \, da^r + \int_{B^r} (\tilde{b}_i * \delta u_i + \tilde{E}_{ij}^A * \delta S_{ij} - \tilde{S}_{ij}^A * \delta E_{ij}) J \, dv^r$$

$$- \frac{1}{\theta_0} * \int_{A_\vartheta^r} \tilde{\vartheta}^p * \delta q^s K \, da^r + \frac{1}{\theta_0} * \int_{A_q^r} \tilde{q}^p * \delta \vartheta K \, da^r - \frac{1}{\theta_0} * \int_{A_c^r} \tilde{\vartheta} * h \delta \vartheta K \, da^r$$

$$\left. - \frac{1}{\theta_0} \int_{B^r} \tilde{\mathscr{R}} * \delta \vartheta J \, dv^r - \frac{1}{\theta_0} * \int_{B^r} (-\tilde{g}_i^A * \delta q_i + \tilde{q}_i^A * \delta g_i) J \, dv^r \right] d\tau$$

$$+ \int_{B^r} (\tilde{u}_i^0 \rho \delta \dot{u}_i|_{(\mathbf{x},0)} + \tilde{v}_i^0 \rho \delta u_i|_{(\mathbf{x},0)}) J \, dv^r. \tag{97}$$

If the adjoint data are specified by eqns (50)–(62), then the right-hand side of this expression is equal to the integrands which contain the implicit response variations in δG_X.

Thus, the left-hand side of eqn (97) replaces the implicit terms in eqn (66) to give the explicit sensitivity:

$$\delta G_X = \int_{\partial B'} s_i * \tilde{u}_i \delta K \, da' - \int_{B'} (S_{ij} * \tilde{u}_{i,m} + \tilde{S}_{ij} * u_{i,m}) \delta J_{mj}^{-1} J \, dv' - \int_{B'} (S_{ij} * \tilde{E}_{ij}$$
$$+ (\rho \ddot{u}_i - b_i) * \tilde{u}_i) \delta J \, dv' + \frac{1}{\theta_0} * \int_{\partial B'} q^s * \tilde{\vartheta} \delta K \, da' - \frac{1}{\theta_0} * \int_{B'} (q_i * \tilde{\vartheta}_{,m}$$
$$+ \tilde{q}_i * \vartheta_{,m}) \delta J_{mi}^{-1} J \, dv' + \frac{1}{\theta_0} \int_{B'} (-1 * q_i * \tilde{g}_i - (c\vartheta - \theta_0 M_{ij} E_{ij} - \mathscr{R}) * \tilde{\vartheta}) \delta J \, dv'$$
$$+ \int_0^t \left[\int_{B'} f \delta J \, dv' + \int_{\partial B'} g \delta K \, da' \right] d\tau. \tag{98}$$

Note that the adjoint systems for the evaluation of δG_D and δG_X are the same, thus only one adjoint problem need be solved.

There are two versions of the material derivative method: the domain and boundary methods. To compare the above result to the one obtained by using the domain version of the material derivative method (Haug *et al.*, 1986), consider the following mapping $\mathbf{x} = \mathbf{r} + \kappa \mathbf{V}(\mathbf{r})$ in which the reference configuration coincides with the current real configuration, i.e. $B = B'$; κ is a time-like parameter, and $\mathbf{V}$ is a fictitious shape design velocity field. Ultimately, δG_X is expressed in terms of the explicitly defined field $\mathbf{V}$, which is viewed as an instantaneous velocity field that defines a deformation-like variation of the current design geometry. At the current design, $B = B'$ and $\kappa = 0$; so $\mathbf{J}$ is the identity tensor, $\partial(\ )/\partial x_i = \partial(\ )/\partial r_i$, and $J = K = 1$. κ is the variation parameter in the expression for $\mathbf{x}$. Therefore, $\delta \mathbf{x} = \delta\kappa \mathbf{V}$, $\delta J_{ij} = \delta\kappa V_{i,j}$, $\delta J_{ij}^{-1} = -\delta\kappa V_{i,j}$, and $\delta J = \delta\kappa V_{i,i}$ (Haug *et al.*, 1986). δK is transformed in a similar manner. Only the normal component of the velocity field is retained in the surface integrals of the material derivative formulation which is presented in Haug *et al.* (1986). When applying the finite element method to such formulations, caution should be exercised. Tangential perturbations of nodes might be required to retain mesh regularity during a shape design process. In general, these tangential movements will affect the solution; therefore, they should be considered in the evaluation of δG_X. The boundary version of the material derivative method can lead to inaccuracies when implemented with the finite element method, because it requires the evaluation of response quantities over the boundary. Some of these quantities (e.g. stress and strain) are difficult to accurately compute over the boundary (Haug *et al.*, 1986). Neither the domain version of the material derivative method nor the present domain parameterization method suffer this drawback.

5. UNCOUPLED, COMBINED QUASI-STATIC UNCOUPLED, AND STEADY-STATE PROBLEMS

Sensitivities for the uncoupled, combined quasi-static uncoupled, and steady-state problems are obtained as specializations of the previous results.

In the uncoupled theory, the $\theta_0 M_{ij} E_{ij}$ terms and their time derivatives are neglected. It is also necessary to eliminate the thermo-coupling (the $\tilde{\vartheta}\mathbf{M}$ term) from the adjoint stress constitutive relation. Equations (46) and (47) become

$$1 * i * \int_B \tilde{E}_{ij} * C_{ijkl} \delta E_{kl} \, dv - i * 1 * \int_B \tilde{g}_i * K_{ij} \delta g_j \, dv$$
$$= 1 * i * \int_B \delta E_{ij} * C_{ijkl} \tilde{E}_{kl} \, dv - i * 1 * \int_B \delta g_i * K_{ij} \tilde{g}_j \, dv \tag{99}$$

$$
1*i*\int_{\partial B}\tilde{u}_i*\delta s_i\,da+1*\int_B\tilde{u}_i*(\delta\mathscr{B}_i-\delta\rho u_i)\,dv-1*i*\int_B\tilde{E}_{ij}*(\delta C_{ijkl}E_{kl}+\delta\vartheta M_{ij}
$$

$$
+\vartheta\delta M_{ij})\,dv+i*1*\int_{\partial B}\tilde{\vartheta}*\delta q^s\,da+i*\int_B\tilde{\vartheta}*(\delta c\vartheta-\delta\mathscr{R})\,dv+i*1*\int_B\tilde{g}_i*\delta K_{ij}g_j\,dv
$$

$$
=1*i*\int_{\partial B}\tilde{s}_i*\delta u_i\,da+1*\int_B\tilde{\mathscr{B}}_i*\delta u_i\,dv+1*i*\int_B(\tilde{E}^A_{ij}C_{ijkl}-\tilde{S}^A_{kl})*\delta E_{kl}\,dv
$$

$$
+i*1*\int_{\partial B}\tilde{q}^s*\delta\vartheta\,da-i*\int_B\tilde{\mathscr{R}}*\delta\vartheta\,dv-i*1*\int_B(K_{ij}\tilde{g}^A_j+\tilde{q}^A_i)*\delta g_i\,dv. \tag{100}
$$

Addition of

$$
1*i*\int_B\tilde{E}^A_{ij}*(\delta C_{ijkl}E_{kl}+\delta M_{ij}\vartheta+M_{ij}\delta\vartheta)\,dv-i*1*\int_B g_i\delta K_{ij}*\tilde{g}^A_j\,dv
$$

to each side, manipulation to isolate implicit and explicit variations, substitution of the modified forms of eqns (18)–(45), and three time differentiations give

$$
-\int_{A_u}\tilde{s}_i*\delta u_i^p\,da+\int_{A_s}\tilde{u}_i*\delta s_i^p\,da+\int_B(\tilde{u}_i*(\delta b_i-\delta\rho\ddot{u}_i)+\dot{\tilde{u}}_i|_{(\mathbf{x},t)}\rho\delta v_i^0)\,dv
$$

$$
-\int_B((\tilde{E}_{ij}-\tilde{E}^A_{ij})*(\vartheta\delta M_{ij}+\delta C_{ijkl}E_{kl}))\,dv-\int_{A_\vartheta}\tilde{q}^s*\delta\vartheta^p\,da+\int_{A_q}\tilde{\vartheta}*\delta q^p\,da
$$

$$
+\int_{A_c}\tilde{\vartheta}*(\delta h(\vartheta-\vartheta_\infty)-h\delta\vartheta_\infty)\,da+\int_B(\tilde{\vartheta}*(\delta c\dot{\vartheta}-\delta r)-\tilde{\vartheta}|_{(\mathbf{x},t)}c\delta\vartheta^0)\,dv
$$

$$
+\int_B(\tilde{g}_i-\tilde{g}^A_i)*\delta K_{ij}g_j\,dv=\int_0^t\Big[-\int_{A_u}\tilde{u}_i^p*\delta s_i\,da+\int_{A_s}\tilde{s}_i^p*\delta u_i\,da+\int_B(\tilde{b}_i*\delta u_i
$$

$$
+\tilde{E}^A_{ij}*\delta S_{ij}-\tilde{S}^A_{ij}*\delta E_{ij})\,dv-\int_{A_\vartheta}\tilde{\vartheta}^p*\delta q^s\,da+\int_{A_q}\tilde{q}^p*\delta\vartheta\,da-\int_{A_c}\tilde{\vartheta}*h\delta\vartheta\,da
$$

$$
-\int_B(\tilde{r}-(\tilde{E}_{ij}-\tilde{E}^A_{ij})M_{ij})*\delta\vartheta\,dv-\int_B(-\tilde{g}^A_i*\delta q_i+\tilde{q}^A_i*\delta g_i)\,dv\Big]d\tau+\int_B(\tilde{u}_i^0\rho\delta\dot{u}_i|_{(\mathbf{x},t)}
$$

$$
+\tilde{v}_i^0\rho\delta u_i|_{(\mathbf{x},t)}-\tilde{\vartheta}^0c\delta\vartheta|_{(\mathbf{x},t)})\,dv. \tag{101}
$$

The adjoint data are again defined to annihilate the response variations in δG_p:

$$
\tilde{b}_i(\mathbf{x},t-\tau)=f_{,u_i}|_{(\mathbf{x},\tau)}\quad\text{in}\quad B\times[0,t] \tag{102}
$$

$$
\tilde{S}^A_{ij}(\mathbf{x},t-\tau)=-f_{,E_{ij}}|_{(\mathbf{x},\tau)}\quad\text{in}\quad B\times[0,t] \tag{103}
$$

$$
\tilde{E}^A_{ij}(\mathbf{x},t-\tau)=f_{,S_{ij}}|_{(\mathbf{x},\tau)}\quad\text{in}\quad B\times[0,t] \tag{104}
$$

$$
\tilde{r}(\mathbf{x},t-\tau)=-f_{,\vartheta}|_{(\mathbf{x},\tau)}+(\tilde{E}_{ij}(\mathbf{x},t-\tau)-\tilde{E}^A_{ij}(\mathbf{x},t-\tau))M_{ij}(\mathbf{x})\quad\text{in}\quad B\times[0,t] \tag{105}
$$

$$
\tilde{q}^A_i(\mathbf{x},t-\tau)=-f_{,g_i}|_{(\mathbf{x},\tau)}\quad\text{in}\quad B\times[0,t] \tag{106}
$$

$$
\tilde{g}^A_i(\mathbf{x},t-\tau)=f_{,q_i}|_{(\mathbf{x},\tau)}\quad\text{in}\quad B\times[0,t] \tag{107}
$$

$$\tilde{u}^p(\mathbf{x}, t-\tau) = -g_{;s_i}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_u \times [0, t] \tag{108}$$

$$\tilde{s}^p(\mathbf{x}, t-\tau) = g_{,u_i}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_s \times [0, t] \tag{109}$$

$$\tilde{\vartheta}^p(\mathbf{x}, t-\tau) = -g_{;q^s}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_\vartheta \times [0, t] \tag{110}$$

$$\tilde{q}^p(\mathbf{x}, t-\tau) = g_{;\vartheta}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_q \times [0, t] \tag{111}$$

$$\tilde{\vartheta}_\infty(\mathbf{x}, t-\tau) = -\frac{1}{h} g_{;\vartheta}|_{(\mathbf{x},\tau)} \quad \text{on} \quad A_C \times [0, t] \tag{112}$$

$$\tilde{u}_i^0(\mathbf{x}) = 0 \quad \text{in} \quad B \quad \text{at} \quad \tau = 0 \tag{113}$$

$$\tilde{v}_i^0(\mathbf{x}) = 0 \quad \text{in} \quad B \quad \text{at} \quad \tau = 0 \tag{114}$$

$$\tilde{\vartheta}^0(\mathbf{x}) = 0 \quad \text{in} \quad B \quad \text{at} \quad \tau = 0. \tag{115}$$

Special care must be taken in solving the adjoint system because the adjoint strain is present in the definition of $\tilde{r}$. First, the adjoint elastic response ($\tilde{\mathbf{u}}$, $\tilde{\mathbf{E}}$, $\tilde{\mathbf{S}}$, and $\tilde{\mathbf{s}}$) is determined. Next, the adjoint strain and the functional definition are used to define the adjoint thermal data; and the adjoint thermal response ($\tilde{\vartheta}$, $\tilde{\mathbf{g}}$, $\tilde{\mathbf{q}}$, and $\tilde{q}^s$) is determined.

The left-hand side of eqn (101) replaces the implicit response variations in eqn (17) to form the explicit sensitivity:

$$\begin{aligned}
\delta G_D = \int_0^t \Bigg(\int_B & (f_{;C_{ijkl}}\delta C_{ijkl} + f_{;\rho}\delta\rho + f_{;K_{ij}}\delta K_{ij} + f_{;c}\delta c + f_{;M_{ij}}\delta M_{ij} \\
& + f_{;\theta_0}\delta\theta_0 + f_{;b_i}\delta b_i + f_{;r}\delta r)\,\mathrm{d}v + \int_{A_u} g_{;u_i^p}\delta u_i^p\,\mathrm{d}a + \int_{A_s} g_{;s_i^p}\delta s_i^p\,\mathrm{d}a \\
& + \int_{A_\vartheta} g_{;\vartheta^p}\delta\vartheta^p\,\mathrm{d}a + \int_{A_q} g_{;q^p}\delta q^p\,\mathrm{d}a + \int_{A_C} (g_{;h}\delta h + g_{;\vartheta_\infty}\delta\vartheta_\infty)\,\mathrm{d}a \Bigg)\,\mathrm{d}\tau \\
& - \int_{A_u} \tilde{s}_i * \delta u_i^p\,\mathrm{d}a + \int_{A_s} \tilde{u}_i * \delta s_i^p\,\mathrm{d}a + \int_B (\tilde{u}_i * (\delta b_i - \delta\rho\ddot{u}_i) + \dot{\tilde{u}}_i|_{(\mathbf{x},t)}\rho\delta u_i^0 \\
& + \tilde{u}_i|_{(\mathbf{x},t)}\rho\delta v_i^0)\,\mathrm{d}v - \int_B ((\tilde{E}_{ij} - \tilde{E}_{ij}^A) * \vartheta\delta M_{ij} + (\tilde{E}_{ij} - \tilde{E}_{ij}^A) * \delta C_{ijkl}E_{kl})\,\mathrm{d}v \\
& - \int_{A_\vartheta} \tilde{q}^s * \delta\vartheta^p\,\mathrm{d}a + \int_{A_q} \tilde{\vartheta} * \delta q^p\,\mathrm{d}a + \int_{A_C} \tilde{\vartheta} * (\delta h(\vartheta - \vartheta_\infty) - h\delta\vartheta_\infty)\,\mathrm{d}a \\
& + \int_B \tilde{\vartheta} * (\delta c\dot{\vartheta} - \delta r - \tilde{\vartheta}|_{(\mathbf{x},t)}c\delta\vartheta^0 - \delta M_{ij}F_{ij})\,\mathrm{d}v \\
& + \int_B (\tilde{g}_i - \tilde{g}_i^A) * \delta K_{ij}g_j\,\mathrm{d}v.
\end{aligned} \tag{116}$$

The same adjoint system is used to obtain δG_X:

$$\delta G_3 = \int_{\partial B'} s_i * \tilde{u}_i \delta K \, da' - \int_{B'} (S_{ij} * \tilde{u}_{i,m} + \tilde{S}_{ij} * u_{i,m}) \delta J_{mj}^{-1} J \, dv' - \int_{B'} (S_{ij} * \tilde{E}_{ij}$$
$$+ (\rho \ddot{u}_i - b_i) * \tilde{u}_i) \delta J \, dv' + \int_{\partial B'} q' * \tilde{\vartheta} \delta K \, da' - \int_{B'} (q_i * \tilde{\vartheta}_{,m}$$
$$+ \tilde{q}_i * \vartheta_{,m}) \delta J_{mi}^{-1} J \, dv' + \int_{B'} (-q_i * \tilde{g}_i + (c\tilde{\vartheta} - r)) * \tilde{\theta} \delta J \, dv' + \int_0^t \left[\int_{B'} f \delta J \, dv' \right.$$
$$\left. + \int_{\partial B'} g \delta K \, da' \right] d\tau. \tag{117}$$

The combined quasi-static uncoupled theory introduces additional simplifications. The inertial terms are neglected in the motion equations to eliminate the displacement time derivatives from all equations. Time is no longer a parameter in a steady-state analysis; so all terms which contain time derivatives are omitted and all operations which involve time integration, differentiation, and convolution are dropped.

If a quasi-static or steady-state boundary-value problem is a traction problem, i.e. $A_u = 0$, then the design functional must be defined to ensure global equilibrium of the adjoint system. Likewise, for steady-state problems, if $A_\vartheta = 0$ and $A_C = 0$, f and g must be defined to ensure a global energy balance for the adjoint system. Fortunately, such problems are seldom encountered.

The sensitivity formulations presented in this subsection differ in some respects from those obtained by Dems (1986) for uncoupled dynamic thermoelasticity and Meric (1986, 1988) for steady-state thermoelasticity. Here, the domain parameterization method is used to derive shape variations; and the convolution is used rather than a time mapping for the transient problem.

These results can be specialized for transient or steady-state linear elasticity or conduction problems to obtain sensitivity expressions consistent with those presented in Dems and Mroz (1983, 1985), Dems (1986), Haber (1987), Meric (1986), Phelan and Haber (1989) and Tortorelli *et al.* (1989, 1990). In Dems and Mroz (1984, 1987), the material derivative approach is used to obtain shape sensitivities, rather than domain parameterization.

6. EXAMPLE

In this section, design sensitivity calculations for a thermoelastic system subjected to transient elastic and thermal loading are presented. The sensitivities are computed using the adjoint load method presented above, and also using the finite difference method for the purpose of verification. The example studies the start-up response of a four-cylinder automotive engine. Sensitivities are computed for performance functionals corresponding to displacement components at selected nodes and the von Mises effective stress at an element Gauss point. Sensitivities with respect to variations in the cylinder wall thickness and the heat transfer coefficients on the inside (gaseous) and the outside (coolant) of the cylinders are computed.

The simulation uses a two-dimensional model of the engine (see Fig. 1), which consists of 450 eight-node, 3×3 integration, isoparametric quadrilateral elements and 1583 nodes. Symmetry about the x-axis is invoked and plane-strain elements are used to model the elastic problem. Node 719 is fixed to prevent rigid body motion. The elastic problem is modeled as a quasi-static system and the thermal problem is modeled as a fully transient system.

The start of the power stroke coincides with the zero degree position of the crank. The gas pressure and the gas temperature inside the cylinders are a function of the crank angle as shown in Figs 2 and 3, respectively. The temperature of the coolant on the outside of the cylinders is assumed to be constant at room temperature (293.16 K) throughout the

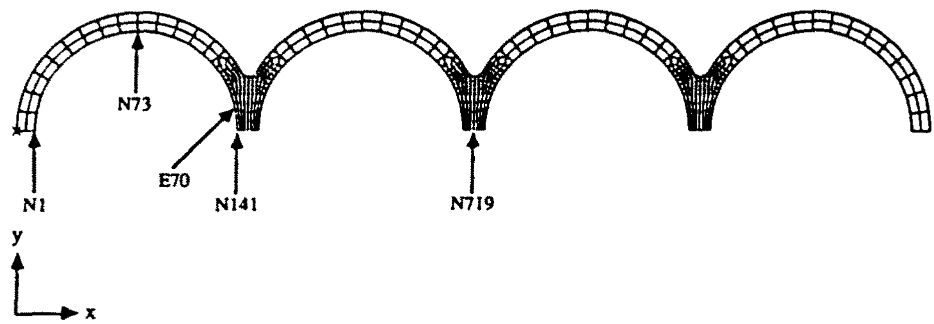

Fig. 1. Finite element model of a cross-section of a four-cylinder automotive engine.

engine start-up. The initial temperature of the engine is also room temperature. Convective boundary conditions are applied to the thermal problem in which the convective coefficients on the coolant side and the gaseous side are 7040 and 116 W m^{-2} K^{-1}, respectively. The operating speed of the engine is 3000 rpm and the cylinder firing order is 1-3-4-2.

The material properties of the current design are those of cast iron: the modulus of elasticity is 103.4 GPa; the Poisson's ratio is 0.25; the thermal conductivity is 29.0 W m^{-1} K^{-1}; the mass density is 7196.6 kg m^{-3}; the specific heat is 440 J kg^{-1} K^{-1}; and the coefficient of thermal expansion is 0.0000129 m m^{-1} K^{-1}.

The sensitivity functionals were defined to represent the distortion of the cylinders and the stresses in the engine during operation. An analysis was performed for a total time of 0.0525 s (which corresponds to a crack rotation of 945°) with a time step $\Delta\tau = 0.000278$ s (corresponding to a crank rotation of 5°) to identify the point of maximum von Mises effective stress during the operation of the engine. The analysis showed that the highest von Mises stress occurs in element 70 at the Gauss point with parametric coordinates $(-0.7746, 0.7746)$ at time $\tau = 0.000833$ s. This time also corresponds to the maximum gas pressure.

A Dirac delta function located at the critical Gauss point was used to localize the von Mises stress performance functional for the sensitivity calculation. In addition, the sensitivity of the x-displacements at nodes 1 and 141, and the y-displacement at node 73 (which are representative of the distortion of the cylinder) were calculated using functionals containing Dirac delta functions at the appropriate nodes. It is difficult to incorporate Dirac delta functions in the time domain in numerical integration schemes. Therefore, an approximation was used in which the functionals are sampled over a single time step and the normalized by the length of the step. The performance functionals used in the sensitivity analyses are:

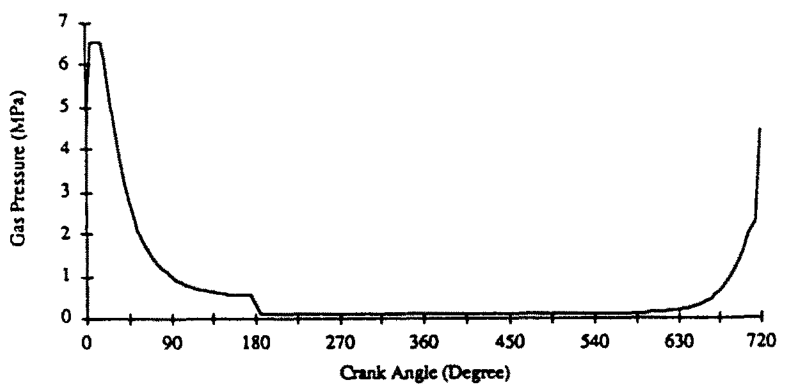

Fig. 2. Gas pressure inside a cylinder vs crank angle.

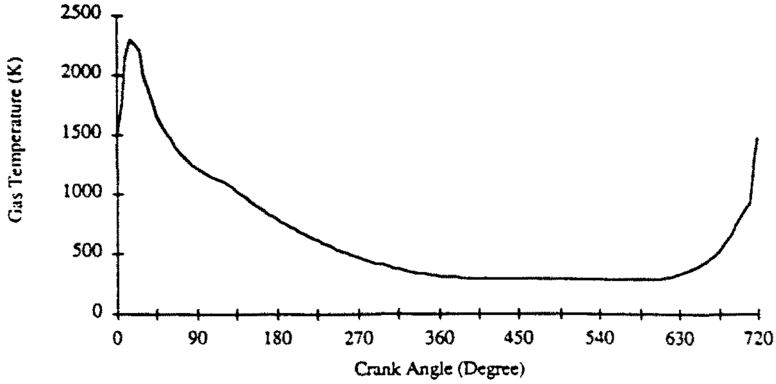

Fig. 3. Gas temperature inside a cylinder vs crank angle.

$$G_1 = \frac{1}{\Delta\tau}\int_{t_s}^{t_f}\int_B \delta(\mathbf{x}-\mathbf{x}_1)u_1 \, \mathrm{d}v \, \mathrm{d}\tau \tag{118}$$

$$G_2 = \frac{1}{\Delta\tau}\int_{t_s}^{t_f}\int_B \delta(\mathbf{x}-\mathbf{x}_{73})v_{73} \, \mathrm{d}v \, \mathrm{d}\tau \tag{119}$$

$$G_3 = \frac{1}{\Delta\tau}\int_{t_s}^{t_f}\int_B \delta(\mathbf{x}-\mathbf{x}_{141})u_{141} \, \mathrm{d}v \, \mathrm{d}\tau \tag{120}$$

$$G_4 = \frac{1}{\Delta\tau}\int_{t_s}^{t_f}\int_B \delta(\mathbf{x}-\mathbf{x}_p)\bar{s}_p \, \mathrm{d}v \, \mathrm{d}\tau \tag{121}$$

where $\mathbf{x}_n$, u_n, and v_n refer to the coordinates and the x- and y-displacements of node n; and $\delta(\mathbf{x}-\mathbf{x}_p)\bar{s}_p$ refers to the von Mises stress at the Gauss point in element 70. The time interval over which the functionals are sampled is given by t_s and t_f, where $t_s = 0.000556$ s and $t_f = 0.0008333$ s. The fraction $(1/\Delta\tau)$ is the normalization factor for this interval. The values of the functionals are -0.02405 mm, 0.01885 mm, -0.00779 mm, and 45.56755 N mm^{-2}, respectively.

The adjoint loads for each of these functionals are defined through eqns (102)–(115). For example, for functional G_1, the adjoint load set for the elastic analysis is given by:

$$\tilde{b}_1(t-\tau) = \begin{cases} \delta(\mathbf{x}-\mathbf{x}_1)/\Delta\tau & \text{if} \quad \tau \in [0.000278, 0.000556] \\ 0 & \text{otherwise}. \end{cases}$$

The adjoint thermal load set is defined by:

$$\tilde{r}(\mathbf{x}, t-\tau) = \tilde{E}_{ij}(\mathbf{x}, t-\tau)M_{ij}(\mathbf{x}) \quad \text{in} \quad B \times [0, t].$$

All other adjoint loads are zero.

The variations with respect to the wall thickness are found for all the above functionals by moving the nodes on the inner surface of the cylinders radially inward. The explicit sensitivity expressions require one adjoint solution for each of the four functionals in addition to the actual solution for the current design. As previously mentioned, the adjoint solutions are found by assembling the adjoint load vectors and peforming back-substitutions on the existing decomposed stiffness matrices (one for the thermal analysis and one for the elastic) which were used for the real analyses. The sensitivities with respect to variations of each node coordinate are then computed as discussed in Section 4. Finally, the variation of each functional G due to perturbations in wall thickness is determined by the chain rule

$$\delta G = \frac{\partial G}{\partial R}\delta R = \frac{\partial G}{\partial X_\alpha}\frac{\partial X_\alpha}{\partial R}\delta R \tag{122}$$

where R refers to the wall thickness and X_α are elements of the global node coordinate vector that lie on the inner surface of the cylinders.

In a similar fashion, the variation of each functional G due to changes of the convection coefficients is found from

$$\delta G = \frac{\partial G}{\partial h_\alpha}\delta h_\alpha \tag{123}$$

where h_α; $\alpha = 1, 2$ are the convection coefficients of the outer and inner walls, respectively. Note that the adjoint solutions required to compute each $\partial G/\partial X_\alpha$ are also used for the calculation of the corresponding $\partial G/\partial h_\alpha$.

To verify the adjoint sensitivity calculations, finite difference sensitivities were also computed. The finite difference shape sensitivities are given by

$$\frac{\partial G}{\partial R} = \frac{G(R+\Delta R)-G(R)}{\Delta R} \tag{124}$$

where ΔR represents a perturbation in the wall thickness. The finite difference sensitivities with respect to the convection coefficients are computed in a similar manner. Each finite difference sensitivity calculation requires one additional real analysis. Thus, the finite difference sensitivities require considerably more computations than the adjoint sensitivities, as the stiffness matrix needs to be reassembled and decomposed. In general, a range of values for ΔR should be tested to ensure that reliable results are obtained (thus further increasing the computational expense). As seen in Tortorelli and Haber (1989) large and small magnitudes of ΔR lead to truncation and round-off errors, respectively.

The sensitivities of the four functionals with respect to variations in wall thickness are shown in Table 1. The results from finite difference sensitivity calculations for six values of ΔR are also included in the table. Similarly, the adjoint sensitivities with respect to variations in heat transfer coefficients on the coolant side and the gaseous side of the cylinders are shown in Tables 2 and 3, along with their corresponding finite difference results.

As seen from the tables, an increased wall thickness increases the values of G_1 and G_3 and decreases the values of G_2 and G_4. An increase in the value of the outer convective coefficient will increase the value of G_3 and decrease the remaining functional values. Increases in the value of the inner convective coefficient will increase G_2 and decrease the other functionals. By comparing the magnitudes of the sensitivities, it is seen that the inner convective coefficient is the most influential, followed by the wall thickness and then the outer convective coefficient. Finally, it is noted that in all cases the finite difference sensitivities are in excellent agreement with the adjoint sensitivities.

Table 1. Sensitivities due to variation in wall thickness

		$\frac{\partial G_1}{\partial R}\times 10^{-3}$	$\frac{\partial G_2}{\partial R}\times 10^{-3}$	$\frac{\partial G_3}{\partial R}\times 10^{-3}$	$\frac{\partial G_4}{\partial R}$
Adjoint		3.9952296	−3.4186460	1.0334289	−8.7993684
Finite difference	$\Delta R = 0.1e+00$	3.9343140	−3.3970654	1.1147598	−8.6988060
	$\Delta R = 0.1e-01$	3.9890376	−3.4165872	1.0420779	−8.7901236
	$\Delta R = 0.1e-02$	3.9946104	−3.4184412	1.0342990	−8.7984540
	$\Delta R = 0.1e-03$	3.9951684	−3.4186255	1.0335157	−8.7992784
	$\Delta R = 0.1e-04$	3.9952260	−3.4186438	1.0334379	−8.7993612
	$\Delta R = 0.1e-05$	3.9952224	−3.4186492	1.0334311	−8.7993684

Table 2. Sensitivities due to variations in h on outer surface

		$\frac{\partial G_1}{\partial h} \times 10^{-7}$	$\frac{\partial G_2}{\partial h} \times 10^{-7}$	$\frac{\partial G_3}{\partial h} \times 10^{-7}$	$\frac{\partial G_4}{\partial h} \times 10^{-4}$
Adjoint		−2.1521062	−3.0618172	1.1305861	−1.1066446
Finite difference	$\Delta h = 0.1e+02$	−2.1513528	−3.0607534	1.1301991	−1.1062627
	$\Delta h = 0.1e-03$	−2.1513528	−3.0607534	1.1301991	−1.1062627
	$\Delta h = 0.1e-04$	−2.1518722	−3.0617643	1.1307438	−1.1066468
	$\Delta h = 0.1e-05$	−2.1506425	−3.0614436	1.1313982	−1.1062868
	$\Delta h = 0.1e-06$	−2.1365812	−3.0596482	1.1426749	−1.1042730
	$\Delta h = 0.le-07$	−2.0034021	−3.0260674	1.1208708	−1.0757019

Table 3. Sensitivities due to variations in h on inner surface

		$\frac{\partial G_1}{\partial h} \times 10^{-2}$	$\frac{\partial G_2}{\partial h} \times 10^{-2}$	$\frac{\partial G_3}{\partial h} \times 10^{-2}$	$\frac{\partial G_4}{\partial h} \times 10^{-3}$
Adjoint		−5.7166388	3.9972960	−2.4497380	−1.5708049
Finite difference	$\Delta h = 0.1e+02$	−5.7206916	3.9893220	−2.4421287	−1.5645337
	$\Delta h = 0.1e-03$	−5.7171348	3.9964968	−2.4489763	−1.5701914
	$\Delta h = 0.1e-04$	−5.7167784	3.9972168	−2.4496617	−1.5707433
	$\Delta h = 0.1e-05$	−5.7167424	3.9972888	−2.4497305	−1.5707988
	$\Delta h = 0.1e-06$	−5.7167388	3.9972960	−2.4497373	−1.5708042
	$\Delta h = 0.1e-07$	−5.7167388	3.9972960	−2.4497370	−1.5708045

7. CONCLUSION

Design sensitivities have been formulated for the linear, dynamic, thermoelastic problem. The variation of a general performance functional was determined with respect to variations in the explicit design fields including shape. In addition to the fully coupled problem, sensitivities for dynamic-uncoupled, quasi-static-uncoupled, and steady-state problems were also derived. In all cases, the reciprocal theorem was used to derive the sensitivities, the convolution operator was implemented to incorporate transients, and domain parameterization was used to describe shape variations.

In an example problem, the finite element method was used to evaluate the real response, adjoint response, and design sensitivities. The sensitivities of stress and displacement based functionals were found with respect to changes in shape and load data. In all cases, excellent agreement was obtained between the adjoint sensitivity calculations and the computationally expensive finite difference sensitivities. This agreement indicates that accurate sensitivities for the finite element model were obtained. As is always the case, the analyst must ensure that the finite element solution is sufficiently accurate, so that the computed sensitivities are meaningful.

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